2023-24 MATH2048: Honours Linear Algebra II
Homework 1 Solution

Due: 2023-09-15 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. (Friedberg 1.3 Q19) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   Proof. We need to show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   ($\Rightarrow$) Suppose $W_1 \cup W_2$ is a subspace of V. If $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, then there exist $w_1 \in W_1 - W_2$ and $w_2 \in W_2 - W_1$. Since $W_1 \cup W_2$ is a subspace, it is closed under addition, so $w_1 + w_2 \in W_1 \cup W_2$. But $w_1 + w_2 \notin W_1$ and $w_1 + w_2 \notin W_2$, which is a contradiction. Hence, $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   ($\Leftarrow$) If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_2$ or $W_1 \cup W_2 = W_1$, respectively. Since W_1 and W_2 are both subspaces of V, it follows that $W_1 \cup W_2$ is a subspace of V. □

2. (Friedberg 1.3 Q21) Consider V as the set of sequences $\{a_n\}$ of real numbers. By Exercise 20 of Section 1.2, it is a vector space over $\mathbb{R}$ (No need to prove this). Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space V.

   Proof. We need to show that the set of convergent sequences is a subspace of the vector space V of sequences of real numbers. Let S denote the set of convergent sequences.

   (1) The zero sequence is a convergent sequence (it converges to 0), so the zero vector is in S.

(2) If $\{a_n\}$ and $\{b_n\}$ are in S, then their sum $\{a_n + b_n\}$ is also a convergent sequence (it converges to $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$), so S is closed under addition.

(3) If $\{a_n\}$ is in S and $c \in \mathbb{R}$, then the scalar multiple $c\{a_n\}$ is a convergent sequence (it converges to $c \lim_{n\to\infty} a_n$), so S is closed under scalar multiplication.

Since S has these properties, it is a subspace of V. □

3. (Modification of Friedberg 1.3 Q24) Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, ..., a_n) \in F^n : a_1 + a_2 + ... + a_n = 0\}$$
$$W_2 = \{(a_1, a_2, ..., a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

Proof. We need to show that F^n is the direct sum of the subspaces W_1 and W_2.

(1) $W_1 \cap W_2 = \{0\}$. Suppose $(a_1, a_2, ..., a_n) \in W_1 \cap W_2$. Then $a_1 = a_2 = \cdots = a_{n-1} = 0$, and $a_1 + a_2 + ... + a_n = 0$. Then each $a_i = 0$, and $(a_1, a_2, ..., a_n)$ is the zero vector

(2) Any vector in F^n can be expressed as the sum of a vector in W_1 and a vector in W_2. Let $(a_1, a_2, ..., a_n) \in F^n$. We can write it as $(a_1, a_2, ..., a_{n-1}, -(a_1 + ...a_{n-1})) + (0, 0, ..., 0, a_1 + ...a_{n-1})$, where $(a_1, a_2, ..., a_{n-1}, -(a_1 + ...a_{n-1})) \in W_1$ and $(0, 0, ..., 0, a_n) \in W_2$.

Therefore, F^n is the direct sum of W_1 and W_2. □

4. (Friedberg 1.6 Q29a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace W_1+W_2 is finite-dimensional, and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Hint: Start with a basis $\{u_1, u_2, ..., u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, ..., u_k, v_1, v_2, ...v_m\}$ for W_1 and to a basis $\{u_1, u_2, ..., u_k, w_1, w_2, ...w_p\}$ for W_2.

Proof. Let $\{u_1, u_2, ..., u_k\}$ be a basis for $W_1 \cap W_2$. Extend this set to a basis $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m\}$ for W_1 and to a basis $\{u_1, u_2, ..., u_k, w_1, w_2, ..., w_p\}$ for W_2.

Then, $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m, w_1, w_2, ..., w_p\}$ spans $W_1 + W_2$. This is because any vector in $W_1 + W_2$ can be written as a sum of a vector in W_1 and a vector in W_2.

We claim that this set is linearly independent. Suppose that

$$\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i + \sum_{i=1}^{p} c_i w_i = 0.$$

Then, $\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i = -\sum_{i=1}^{p} c_i w_i$ is in $W_1 \cap W_2$ because the left-hand side is in W_1 and the right-hand side is in W_2. But then $-\sum_{i=1}^{p} c_i w_i = \sum_{i=1}^{k} d_i u_i$ for some d_i. Because the vectors $\{u_1, u_2, ..., u_k, w_1, w_2, ..., w_p\}$ are linearly independent, each $c_i, d_i = 0$. Then $\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i = 0$.

But the vectors $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m\}$ are linearly independent, so all the coefficients a_i, b_i must be zero.

Therefore, $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m, w_1, w_2, ..., w_p\}$ is a basis for $W_1 + W_2$, so $W_1 + W_2$ is finite-dimensional and

$$\dim(W_1 + W_2) = k + m + p = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

□

5. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd function if $g(-t) = -g(t)$ for each $t \in F_1$. By Q22 of Section 1.3 the set of all even functions $\mathcal{E}(F_1, F_2)$ in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions $\mathcal{O}(F_1, F_2)$ in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$ (no need to prove this).

   Suppose $2 \neq 0$ in F_2. Show that $\mathcal{F}(F_1, F_2) = \mathcal{E}(F_1, F_2) \oplus \mathcal{O}(F_1, F_2)$. (When $2 = 0$ in F_2, what happens?)

   Proof. Assume $2 \neq 0$ in F_2. We want to show that every function $f \in \mathcal{F}(F_1, F_2)$ can be uniquely written as the sum of an even function and an odd function, i.e., $\mathcal{F}(F_1, F_2) = \mathcal{E}(F_1, F_2) \oplus \mathcal{O}(F_1, F_2)$.

   Given $f \in \mathcal{F}(F_1, F_2)$, define $g, h \in \mathcal{F}(F_1, F_2)$ by $g(t) = \frac{f(t)+f(-t)}{2}$ and $h(t) = \frac{f(t)-f(-t)}{2}$ for each $t \in F_1$. Then, g is even, h is odd, and $f = g + h$, so every function in $\mathcal{F}(F_1, F_2)$ can be written as the sum of an even function and an odd function.

   Suppose $f \in \mathcal{E}(F_1, F_2) \cap \mathcal{O}(F_1, F_2)$. Then $-f(t) = f(-t) = f(t)$ for any t. Then $2f(t) = 0$. Because $2 \neq 0$ in F_2, one can divide by 2, and get $f(t) = 0$. Hence, f is the zero function.

Therefore, $\mathcal{F}(F_1, F_2) = \mathcal{E}(F_1, F_2) \oplus \mathcal{O}(F_1, F_2)$.

When $2 = 0$ in F, we have $-1 = 1$ and odd functions are the same as even functions, and they constitute a subspace of all functions. e.g. $f : \mathbb{R} \to \mathbb{F}_2$ with $f(x) = 1, x \geq 0$, $f(x) = 0, x < 0$. This f is neither odd nor even, and assumes no odd-even decomposition.

Fields where $2 = 0$ are all vector spaces over $\mathbb{F}_2$, and will be studied in MATH2070(2078) and MATH3040. We say that such a field has characteristic 2. A fact by Galois is that there is a finite field with 2^m elements for every integer $m \geq 1$. $\square$